Reliability analysis of $M/G/1$ queues with general retrial times and server breakdowns*

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Received June 21, 2005; revised January 17, 2006

Abstract This paper concerns the reliability issues as well as queueing analysis of $M/G/1$ retrial queues with general retrial times and server subject to breakdowns and repairs. We assume that the server is unreliable and customers who find the server busy or down are queued in the retrial orbit in accordance with a first-come-first-served discipline. Only the customer at the head of the orbit queue is allowed for access to the server. The necessary and sufficient condition for the system to be stable is given. Using a supplementary variable method, we obtain the Laplace-Stieltjes transform of the reliability function of the server and a steady state solution for both queueing and reliability measures of interest. Some main reliability indexes, such as the availability, failure frequency, and the reliability function of the server, are obtained.

Keywords: **retrial queues, general retrial times, reliability, server breakdowns and repairs.**

Retrial queueing systems are characterized by the feature that arrivals who find the server unavailable are obliged to leave the service area and to try again for their requests in random order and at random intervals. Between trials a customer is called to be in "orbit". This feature plays a special role in several computer and communications networks. For recent bibliographies on retrial queues, see the studies by Yang and Templeton[1], Falin[2], Kulkarni and Liang[3], and Falin and Templeton[4]. Artalejo[5,6] also provided extensive surveys of retrial queues.

Many of the retrial queueing systems operate under the classical retrial policy. Nevertheless, there are other types of queueing situations in which the intervals separating successive repeated attempts are independent of the number of the customers in orbit. The so-called constant retrial policy arises naturally in problems where the server is required to search for customers[7] and in communication protocols of type carrier sense multiple access (CSMA). This discipline was introduced by Fayolle[8], who investigated an $M/M/1$ retrial queue in which the repeated customers form a queue and only the customer at the head of the orbit queue can request a service after an exponentially distributed retrial time. Farahmand[9] calls this discipline a retrial queue with FCFS orbit. Since Fayolle[8], there has been a fast development about retrial queues with constant repeated attempts[9—13]. Choi et al.[14] generalized the constant retrial policy by considering an $M/M/1$ retrial queue with general retrial times where only the customer at the head of the orbit may attempt retrials from orbit. Later, Gómez-Corral[15] discussed extensively a retrial queuing system with FCFS discipline and general retrial times. Recently, there have been significant contributions to retrial queues with general service times and non-exponential retrial time distributions. For more information, see Refs. [10, 16—18] and references therein.

On the other hand, most papers on retrial queues also assume that the server is available on a permanent basis. However, in practice, these assumptions are apparently unrealistic. The server may well be subject to lengthy and unpredictable breakdowns while serving a customer. For example, in computer systems, the machine may be subject to scheduled backups and unpredictable failures. Because of the limited ability of repairs and heavy influence of the breakdowns on the performance measure of the system, it is of basic importance to study reliability of retrial queues with server breakdowns and repairs. Wang et al.[19] carried out a detailed analysis of reliability of the classic $M/G/1$ retrial queues with exponentially distributed retrial times. For more information, interested read-

* Supported by Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry, National Natural Science Foundation of China (Grant Nos. 10526004, 60504016), and BJTU Foundation (Grant No. 2005sm064)

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ers may refer to Refs.[20—22], where a single-server retrial queue with unreliable server was considered.

In this paper, we discuss the $M/G/1$ retrial queues with general retrial times and server subject to breakdowns and repairs. The basic model was investigated by Gómez-Corral[15] but without server breakdowns (failure rate $\mu=0$, see Remark 2 in Section 3). In this sense, we generalize the corresponding model by studying the reliability issues as well as queueing characteristics of the unreliable retrial queues. The explicit expressions of some main queueing measures along with the main reliability measures such as availability, failure frequency, and reliability function of the server are obtained.

1 Model description

We consider a single-server retrial queue in which primary customers arrive according to a Poisson process with rate $\lambda>0$. There is no waiting space in front of the server where the server is subject to breakdowns and repairs. If an arriving primary customer finds the server idle, the customer begins service immediately and leaves the system after service completion. If the server is found to be busy or under repair, the arriving primary customer enters a retrial queue according to a FCFS discipline. For customers in the retrial queue, we assume that only the customer at the head of the retrial queue is allowed to attempt to reach the server. When a service is completed, the server searches for customers in the retrial orbit for next service. The searching time is governed by an arbitrary law with common probability distribution function $A(x)[A(0)=0]$, of density function $a(x)$ and Laplace-Stieltjes transform $\tilde{A}(s)$ and measured from the instant the server is idle. During the searching process, if a primary customer arrives at the system, then the server interrupts the searching process and begins to serve the primary customer. Otherwise, the customer at the head of the retrial queue is selected to start his service. That is, when the server becomes idle, a customer from the retrial queue competes with a primary customer to decide who reaches the server first. The time generated by $A(x)$ is called the retrial time. The retrial customer is required to cancel the attempt for service if a primary customer arrives first. In that case, the retrial customer returns to its position in the retrial queue.

The service times are independent, identically distributed random variables with a common probability distribution function $B(x)[B(0)=0]$, density function $b(x)$, Laplace-Stieltjes transform $\tilde{B}(s)$, and first two moments β_1 and β_2. We assume that when the server is busy it fails at an exponential rate μ. When the server fails, repair begins immediately and the customer just being served before server breakdown waits for the server until repair completion in order to complete its remaining service. The repair time is a random variable with probability distribution function $G(x)$, density function $g(x)$, Laplace-Stieltjes transform $\tilde{G}(s)$, and with the first two moments γ_1 and γ_2.

We assume that the service time for a customer is cumulative and server is as good as new after repair. Note that by assumption, the total "service time" is a time interval measured from when a customer begins to be served until the service is completed, which comprises possible breakdowns times. We define this time interval as the "generalized service time". The inter-arrival times of primary calls, retrial times, service times, repair times are assumed to be mutually independent.

The functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are the conditional completion rates for repeated attempt, for service, and for repair at time t, which are defined respectively as

$$\alpha(x)=\frac{a(x)}{1-A(x)},\quad \beta(x)=\frac{b(x)}{1-B(x)},$$
$$\gamma(x)=\frac{g(x)}{1-G(x)}.$$

From the description of the model, the state of the system at time t can be described by the Markov process $\{X(t),\ t\geqslant 0\}=\{(C(t),\ Q(t),\ \xi_0(t),\ \xi_1(t),\ \xi_2(t));\ t\geqslant 0\}$, where at time t, $C(t)$ denotes the server state (0, 1, 2 stand for the server idle, busy, under repair, respectively) and $Q(t)$ denotes the number of customers in the retrial orbit. Three supplementary variables are introduced to analyze the states of the system. At time t, if $C(t)=0$ and $Q(t)>0$, we define $\xi_0(t)$ as the elapsed retrial time; if $C(t)=1$ or $C(t)=2$, we define $\xi_1(t)$ as the elapsed service time; if $C(t)=2$, we define $\xi_2(t)$ as the elapsed repair time. Thus, the stochastic process $\{X(t), t\geqslant 0\}=\{(C(t),\ Q(t),\ \xi_0(t),\ \xi_1(t), \xi_2(t));\ t\geqslant 0\}$ is a Markov process. Fig. 1 illustrates the system behavior.

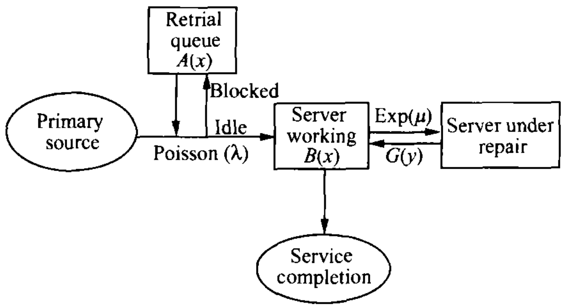

Fig. 1. Description of the system behavior.

2 Stability condition

To prove ergodicity, we need the following preliminary results:

Let χ_n be the generalized service time of the nth customer. Note that it may include some possible down times of the server due to server failures during the service period of the nth customer, since the nth customer begins to be served until the service is completed. Cao and Cheng[23] have found that χ_n's are independent, identically distributed random variables with distribution function

$$D(t) \triangleq Pr\{\chi_n \leqslant t\} = \sum_{l=0}^{\infty}\int_0^t G^{(l)}(t-u)e^{-\mu u}\frac{(\mu u)^l}{l!}dB(u), \quad (1)$$

which is independent of n. Its Laplace-Stieltjes transform is

$$\tilde{D}(s) = \int_0^{\infty} e^{-st}dD(t) = \tilde{B}(s+\mu-\mu\tilde{G}(s)), \quad \mathrm{Re}(s) > 0,$$

and its expected value is given by

$$E\chi_n = -\left.\frac{d\tilde{D}(s)}{ds}\right|_{s=0} = \beta_1(1+\mu\gamma_1).$$

Thus, we can provide the following theorem which gives a necessary and sufficient condition for the system to be stable.

Theorem 1. The inequality $\lambda\beta_1(1+\mu\gamma_1) < \tilde{A}(\lambda)$ is a sufficient and necessary condition for the system to be stable.

Proof. Let Q_n be the number of customers in the retrial queue at the nth departure point, $n \geqslant 1$. It is not difficult to see that $\{Q_n, n\geqslant 1\}$ is an irreducible and aperiodic Markov chain, with the state space Z_+ and the following one-step transition probabilities:

$$P(Q_{n+1}=j \mid Q_n = i) = \begin{cases} \int_0^{\infty}\frac{(\lambda x)^j}{j!}e^{-\lambda x}dD(x), & \text{if } i=0 \text{ and } j\geqslant 0 \\ \tilde{A}(\lambda)\int_0^{\infty}e^{-\lambda x}dD(x), & \text{if } i>0 \text{ and } j=i-1 \\ (1-\tilde{A}(\lambda))\int_0^{\infty}\frac{(\lambda x)^{j-i}}{(j-i)!}e^{-\lambda x}dD(x) & \\ \quad +\tilde{A}(\lambda)\int_0^{\infty}\frac{(\lambda x)^{j-i+1}}{(j-i+1)!}e^{-\lambda x}dD(x), & \text{if } i>0 \text{ and } j>i-1 \\ 0, & \text{otherwise} \end{cases}. \quad (2)$$

We first prove that $\{Q_n, n\geqslant 1\}$ is ergodic. To this end, we shall use the following Foster's criterion: An irreducible and aperiodic Markov chain $\{Q_n, n\geqslant 0\}$ is ergodic if there exists a nonnegative function $f(j)$, $j\in\mathbb{N}$, and $\varepsilon>0$ such that the mean drift $\Psi_k = E[f(Q_{n+1})-f(Q_n)\mid Q_n=k]$ satisfies $|\Psi_k|<\infty$ for all $k\in\mathbb{N}$, except perhaps a finite number of j. We consider the function $f(j)=j$ in our case. Thus, we obtain that $\Psi_k = \lambda\beta_1(1+\mu\gamma_1) - (1-\delta_{0k})\tilde{A}(\lambda)$ for $k\geqslant 0$, where δ_{ij} denotes Kronecker's delta. Clearly, the inequality $\lambda\beta_1(1+\mu\gamma_1) < \tilde{A}(\lambda)$ is a sufficient condition for the system to be stable.

The same inequality is also necessary for ergodicity. We prove this by employing a theorem proposed by Sennot et al.[24], which states that the Markov chain $\{Q_n, n\geqslant 0\}$ is not ergodic if it satisfies Kaplan's condition, that is, the mean drift $\Psi_k<\infty$ $(k\geqslant 0)$, and there exists $K\in\mathbb{N}$ such that $\Psi_k\geqslant 0$ for $k\geqslant K$.

Let $\{Q_n, n\geqslant 0\}$ be ergodic and assume that $\lambda\beta_1(1+\mu\gamma_1)\geqslant\tilde{A}(\lambda)$. From the above discussion, $\Psi_k\geqslant 0$ $(k\geqslant 0)$. However, according to (2), the down drift $\mathscr{D}_i \equiv \sum_{j<i}(j-i)P(Q_{n+1}=j\mid Q_n=i)$ is equal to 0 if $i=0$ and equal to

$$- \widetilde{A}(\lambda)\int_0^{\infty} e^{-\lambda x} dD(x) \text{ if } i > 0.$$

This implies that Kaplan's condition holds by Theorem 3 in Ref. [24], which states that Kaplan's condition holds if the sequence $\{\mathscr{D}_i\}$ is bounded below. Thus, $\{Q_n, n \geqslant 0\}$ is not ergodic which is contradictory. Hence, the necessity of the ergodicity is proven.

Since primary customers arrive in according to a Poisson process, it is well known that the steady state probabilities of $\{(C(t), Q(t)); t \geqslant 0\}$ exist and are positive if and only if $\{Q_n, n \geqslant 0\}$ is ergodic. Therefore, it suffices to show that $\lambda\beta_1(1+\mu\gamma_1) < \widetilde{A}(\lambda)$ is a sufficient and necessary condition for the system to be stable. Q.E.D.

Remark 1. To explain the one-step transition probabilities in (2), we consider the following cases:

(i) When $i=0$, there is no customer in the retrial orbit at the moment that the nth customer leaves the system. If there are j customers in the orbit when the $(n+1)$th customer leaves the system, we know that these customers arrive at the system during the generalized service time of the $(n+1)$th customer. Then, we get the first expression in (2) due to the Poisson arrival process.

(ii) When $i>0$ and $j=i-1$, there are i customers in the orbit when the nth customer leaves the system, and the next customer who enters the service is from retrial orbit (with probability $\widetilde{A}(\lambda)$). Furthermore, there is no external arrival during the service (with probability $\int_0^{\infty} e^{-\lambda x} dD(x)$). Combining these factors, we have the second expression in (2).

(iii) When $i>0$ and $j>i-1$, there are i customers in the orbit when the nth customer leaves the system, and the next customer who enters the service can be: (a) primary customer with probability $1-\widetilde{A}(\lambda)$, and during this service there are $j-i$ customers arriving at the system; (b) retrial customer with probability $\widetilde{A}(\lambda)$ and during this service there are $j-i+1$ customers arriving at the system. By total probability law, we obtain the third expression.

(iv) Other events are impossible, so the corresponding one-step transition probabilities are zero.

3 The steady state equations and solutions

In this section, we shall study the system in the steady state which exists if and only if the stability condition is met.

For the process $\{X(t), t \geqslant 0\}$, we define the state probability $p_{00}(t) \equiv P(C(t)=0, Q(t)=0)$ and the state probability densities $p_{0,i}(t,v)dv \equiv P(C(t)=0, Q(t)=i, v<\xi_0(t)<v+dv)$, for $i \geqslant 1$; $p_{1,i}(t,x)dx \equiv P(C(t)=1, Q(t)=i, x<\xi_1(t)<x+dx)$, for $i \geqslant 0$; $p_{2,i}(t,x,y)dxdy \equiv P(C(t)=2, Q(t)=i, x<\xi_1(t)<x+dx, y<\xi_2(t)<y+dy)$, for $i \geqslant 0$.

In a general way we obtain the equations of statistical equilibrium:

$$\frac{d}{dt}p_{00}(t) + \lambda p_{00}(t) = \int_0^{\infty} p_{1,0}(t,x)\beta(x)dx, \tag{3}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial v} + \lambda + \alpha(v)\right]p_{0,j}(t,v) = 0, \quad j \geqslant 1, \tag{4}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \mu + \beta(x)\right]p_{1,j}(t,x) = (1-\delta_{0j})\lambda p_{1,j-1}(t,x) + \int_0^{\infty}\gamma(y)p_{2,j}(t,x,y)dy, j \geqslant 0, \tag{5}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \lambda + \gamma(y)\right]p_{2,j}(t,x,y) = \lambda p_{2,j-1}(t,x,y), \quad j \geqslant 0. \tag{6}$$

The above equations are to be solved under the boundary conditions

$$p_{0,j}(t,0) = \int_0^{\infty} p_{1,j}(t,x)\beta(x)dx, \tag{7}$$

$$p_{1,j}(t,0) = \lambda\delta_{0j}p_{00}(t) + (1-\delta_{0j})\int_0^{\infty} p_{0,j}(t,v)dv + \int_0^{\infty} P_{0,j+1}(t,v)\alpha(v)dv, \tag{8}$$

$$p_{2,j}(t,x,0) = \mu p_{1,j}(t,x), \tag{9}$$

and the normalization equation

$$p_{00}(t) + \sum_{i=1}^{\infty}\int_0^{\infty} p_{0,j}(t,v)dv + \sum_{i=0}^{\infty}\left[\int_0^{\infty} p_{1,j}(t,x)dx + \int_0^{\infty}\int_0^{\infty} p_{2,j}(t,x,y)dxdy\right] = 1,$$

where δ_{0j} is the Kronecker function, $p_{1,-1}(t,x)\equiv 0$, $p_{2,-1}(t,x,y)\equiv 0$ for $0\leqslant t,x,y<\infty$.

We assume that the condition $\lambda\beta_1(1+\gamma_1)<\tilde{A}(\lambda)$ is fulfilled and set $p_{00}=\lim_{t\to\infty}p_{00}(t)$, $p_{0,j}(v)=\lim_{t\to\infty}p_{0,j}(t,v)$ for $j\geqslant 1$ and $v\geqslant 0$; $p_{1,j}(x)=\lim_{t\to\infty}p_{1,j}(t,x)$ for $j\geqslant 0$ and $x\geqslant 0$; and $p_{2,j}(x,y)=\lim_{t\to\infty}p_{2,j}(t,x,y)$ for $x\geqslant 0$, $y\geqslant 0$, $j\geqslant 0$.

Theorem 2. In the steady state, the joint distribution of the server state and the length of the retrial queue has partial generating function:

$$p_{00}=1-\frac{\lambda\beta_1(1+\mu\gamma_1)}{\tilde{A}(\lambda)},\tag{10}$$

$$P_0(z,v)=\frac{\lambda z[\tilde{A}(\lambda)-\lambda\beta_1(1+\mu\gamma_1)][1-\tilde{B}(\Phi(\lambda-\lambda z))]\times\exp\{-\lambda v\}(1-A(v))}{\tilde{A}(\lambda)\{(1-z)\tilde{A}(\lambda)\tilde{B}[\Phi(\lambda-\lambda z)]-z[1-\tilde{B}(\Phi(\lambda-\lambda z))]\}},\tag{11}$$

$$P_1(z,x)=\frac{\lambda(1-z)[\tilde{A}(\lambda)-\lambda\beta_1(1+\mu\gamma_1)]\times\exp\{-\Phi(\lambda-\lambda z)x\}(1-B(x))}{(1-z)\tilde{A}(\lambda)\tilde{B}[\Phi(\lambda-\lambda z)]-z[1-\tilde{B}(\Phi(\lambda-\lambda z))]},\tag{12}$$

$$P_2(z,x,y)=\frac{\lambda\mu(1-z)[\tilde{A}(\lambda)-\lambda\beta_1(1+\mu\gamma_1)]}{(1-z)\tilde{A}(\lambda)\tilde{B}[\Phi(\lambda-\lambda z)]-z[1-\tilde{B}(\Phi(\lambda-\lambda z))]}\times\exp\{-\Phi(\lambda-\lambda z)x-\lambda(1-z)y\}(1-B(x))(1-G(y)),\tag{13}$$

where $\Phi(x)\equiv x+\mu-\mu\tilde{G}(x)$, and

$$P_0(z,v)=\sum_{j=1}^{\infty}p_{0,j}(v)z^j,$$

$$P_1(z,x)=\sum_{j=0}^{\infty}p_{1,j}(x)z^j,$$

$$P_2(z,x,y)=\sum_{j=0}^{\infty}p_{2,j}(x,y)z^j.$$

Proof. Under the stability condition, we derive from Eqs. (3)—(9) that

$$\lambda p_{00}=\int_0^\infty p_{1,0}(x)\beta(x)\mathrm{d}x,\tag{14}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}v}+\lambda+\alpha(v)\right]p_{0,j}(v)=0,\tag{15}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}+\lambda+\mu+\beta(x)\right]p_{1,j}(x)=\lambda p_{1,j-1}(x)+\int_0^\infty\gamma(y)p_{2,j}(x,y),\tag{16}$$

$$\left[\frac{\partial}{\partial y}+\lambda+\gamma(y)\right]p_{2,j}(x,y)=\lambda p_{2,j-1}(x,y),\tag{17}$$

$$p_{0,j}(0)=\int_0^\infty p_{1,j}(x)\beta(x)\mathrm{d}x,\tag{18}$$

$$p_{1,j}(0)=\lambda\delta_{0j}p_0+(1-\delta_{0j})\int_0^\infty p_{0,j}(v)\mathrm{d}v+\int_0^\infty p_{0,j+1}(v)\alpha(v)\mathrm{d}v,\tag{19}$$

$$p_{2,j}(x,0)=\mu p_{1,j}(x),\tag{20}$$

and the normalization condition

$$p_{00}+\sum_{j=1}^{\infty}\int_0^\infty p_{0,j}(v)\mathrm{d}v+\sum_{j=0}^{\infty}\left[\int_0^\infty p_{1,j}(x)\mathrm{d}x+\int_0^\infty\int_0^\infty p_{2,j}(x,y)\mathrm{d}x\mathrm{d}y\right]=1.\tag{21}$$

Multiplying both sides of Eqs. (15)—(20) by z^j and summing over j, we obtain the following equations after some algebraic manipulations:

$$\left[\frac{\partial}{\partial v}+\lambda+\alpha(v)\right]P_0(z,v)=0,\tag{22}$$

$$\left[\frac{\partial}{\partial x}+\lambda+\mu+\beta(x)\right]P_1(z,x)=\lambda zP_1(z,x)+\int_0^\infty P_2(z,x,y)\gamma(y)\mathrm{d}y,\tag{23}$$

$$\left[\frac{\partial}{\partial y}+\lambda+\gamma(y)\right]P_2(z,x,y)=\lambda zP_2(z,x,y),\tag{24}$$

$$P_0(z,0)=-\lambda p_{00}+\int_0^\infty P_1(z,x)\beta(x)\mathrm{d}x,\tag{25}$$

$$P_1(z,0)=\lambda\left[p_{00}+\int_0^\infty P_0(z,v)\mathrm{d}v\right]+\frac{1}{z}\int_0^\infty P_0(z,v)\alpha(v)\mathrm{d}v,\tag{26}$$

$$P_2(z,x,0)=\mu P_1(z,x).\tag{27}$$

In addition, Eq. (21) can be rewritten as

$$p_{00}+\lim_{z\to 1-}\left[\int_0^\infty P_0(z,v)\mathrm{d}v+\int_0^\infty P_1(z,x)\mathrm{d}x+\int_0^\infty\int_0^\infty P_2(z,x,y)\mathrm{d}x\mathrm{d}y\right]=1.$$

Considering Eq. (27), we obtain the solutions to Eqs. (22), (24) that

$$P_0(z,v)=P_0(z,0)\mathrm{e}^{-\lambda v}(1-A(v)),\tag{28}$$

$$P_2(z,x,y)=P_1(z,x)\mu\mathrm{e}^{-\lambda(1-z)y}(1-G(y)).\tag{29}$$

Substituting Eq. (28) into (26), we get

$$P_1(z,0)=\lambda p_{00}+\frac{1}{z}P_0(z,0)[\tilde{A}(\lambda)+(1-\tilde{A}(\lambda))z].\tag{30}$$

Substituting Eq. (29) into (22), we have

$$P_1(z,x)=P_1(z,0)\mathrm{e}^{-\Phi(\lambda(1-z))x}(1-B(x)),\tag{31}$$

which is substituted into (25) to give

$$P_0(z,0) = -\lambda p_{00} + \tilde{B}(\Phi(\lambda(1-z)))P_1(z,0). \quad (32)$$

Thus, from (30) and (32), we get

$$P_0(z,0) = \frac{\lambda z\,(1-\tilde{B}(\Phi(\lambda(1-z))))}{\tilde{A}(\lambda)(1-z)\tilde{B}(\Phi(\lambda(1-z))) - z(1-\tilde{B}(\Phi(\lambda(1-z))))}p_{00}, \quad (33)$$

$$P_1(z,0) = \frac{\lambda\tilde{A}(\lambda)(1-z)}{\tilde{A}(\lambda)(1-z)\tilde{B}(\Phi(\lambda(1-z))) - z(1-\tilde{B}(\Phi(\lambda(1-z))))}p_{00}, \quad (34)$$

which implies that $P_0(z,v)$, $P_1(z,x)$, $P_2(z,x,y)$ depend only upon p_{00}. Finally, p_{00} can be found by using the normalization condition

$$p_{00} + \int_0^\infty P_0(1,v)\mathrm{d}v + \int_0^\infty P_1(1,x)\mathrm{d}x + \int_0^\infty\int_0^\infty P_2(1,x,y)\mathrm{d}x\mathrm{d}y = 1,$$

which completes the proof. Q.E.D.

Corollary 1. If $\lambda\beta_1(1+\mu\gamma_1) < \tilde{A}(\lambda)$, then

(i) the probability that the system is empty:

$$p_{00} \equiv 1 - \frac{\lambda\beta_1(1+\mu\gamma_1)}{\tilde{A}(\lambda)};$$

(ii) the probability that the server is idle and the system is not empty:

$$P_{\text{idle}} = \frac{(1-\tilde{A}(\lambda))\lambda\beta_1(1+\mu\gamma_1)}{\tilde{A}(\lambda)};$$

(iii) the probability that the server is busy:

$$P_{\text{busy}} = \lambda\beta_1;$$

(iv) the probability that the server is under repair:

$$P_{\text{repair}} = \lambda\beta_1\mu\gamma_1.$$

Corollary 2. In the steady state, denote by N_q and N the number of the customers in the retrial queue and in the system, respectively. Then N_q and N have probability generating functions

$$p_q(z) = \frac{(\tilde{A}(\lambda) - \lambda\beta_1(1+\mu\gamma_1))(1-z)}{\tilde{A}(\lambda)(1-z)\tilde{B}(\Phi(\lambda(1-z))) - z(1-\tilde{B}(\Phi(\lambda(1-z))))}, \quad (35)$$

$$p(z) = \frac{(\tilde{A}(\lambda - \lambda\beta_1(1+\mu\gamma_1))(1-z)\tilde{B}(\Phi(\lambda(1-z))))}{\tilde{A}(\lambda)(1-z)\tilde{B}(\Phi(\lambda(1-z))) - z(1-\tilde{B}(\Phi(\lambda(1-z))))}. \quad (36)$$

In particular, the corresponding expected values are given by

$$L_q = \frac{\lambda^2\mu\gamma_2\beta_1^3 + 2\rho(1-\tilde{A}(\lambda))\beta_1^2 + \rho^2\beta_2}{2(\tilde{A}(\lambda)-\rho)\beta_1^2}, \quad (37)$$

$$L = \frac{\lambda^2\mu\gamma_2\beta_1^3 + 2\rho(1-\rho)\beta_1^2 + \rho^2\beta_2}{2(\tilde{A}(\lambda)-\rho)\beta_1^2}, \quad (38)$$

where $\rho \equiv \lambda\beta_1(1+\mu\gamma_1)$.

Proof. This is readily obtained by considering the following equations:

$$p_q(z) = P_{00} + \int_0^\infty P_0(z,v)\mathrm{d}v + \int_0^\infty P_1(z,x)\mathrm{d}x + \int_0^\infty\int_0^\infty P_2(z,x,y)\mathrm{d}x\mathrm{d}y,$$

$$p(z) = P_{00} + \int_0^\infty P_0(z,v)\mathrm{d}v + z\left[\int_0^\infty P_1(z,x)\mathrm{d}x + \int_0^\infty\int_0^\infty P_2(z,x,y)\mathrm{d}x\mathrm{d}y\right],$$

and the mean queue length can be readily derived from (35) and (36). Q.E.D.

Remark 2. When $\mu = 0$, our model becomes the corresponding $M/G/1$ retrial queue reliable server. In this case, (10)—(12) reduce to

$$P_{00} = 1 - \frac{\lambda\beta_1}{\tilde{A}(\lambda)},$$

$$P_0(z,v) = \frac{\lambda z[\tilde{A}(\lambda) - \lambda\beta_1][1-\tilde{A}(\lambda-\lambda z)]}{\tilde{A}(\lambda)\{(1-z)\tilde{A}(\lambda)\tilde{B}(\lambda-\lambda z) - z[1-\tilde{B}(\lambda-\lambda z)]\}} \times \exp\{-\lambda v\}(1-A(v)),$$

$$P_1(z,x) = \frac{\lambda(1-z)[\tilde{A}(\lambda)-\lambda\beta_1]}{(1-z)\tilde{A}(\lambda)\tilde{B}(\lambda-\lambda z) - z[1-\tilde{B}(\lambda-\lambda z)]} \times \exp\{-(\lambda-\lambda z)x\}(1-B(x)),$$

which agree with (13)—(15) in Gómez-Corral[15].

The following theorem gives the distributions of the waiting times and the corresponding expected values in the steady state.

Theorem 3. Denote by W and W_q, respectively, the time that an arriving primary customer spends in the system and in the retrial queue at the steady state. If the system is stable, then the Laplace-Stieltjes transforms of W and W_q are given by

$$\tilde{W}(s) = (1-\rho)\tilde{B}(\Phi(s)) + \frac{\lambda(\tilde{A}(\lambda)-\rho)}{\lambda - s - \lambda\pi(s)}$$

$$\times \frac{\pi(s)(1-\pi(s))(\widetilde{B}(\Phi(s))-\widetilde{B}(\Phi(\lambda(1-\pi(s)))))}{\widetilde{A}(\lambda)(1-\pi(s))\widetilde{B}(\Phi(\lambda(1-\pi(s)))-\pi(s)(1-\widetilde{B}(\Phi(\lambda(1-\pi(s)))))}, \quad (39)$$

$$\widetilde{W}_q(s) = 1-\rho+\frac{\lambda(\widetilde{A}(\lambda)-\rho)}{(\lambda-s-\lambda\pi(s))\widetilde{B}(\Phi(s))}$$
$$\times \frac{\pi(s)(1-\pi(s))(\widetilde{B}(\Phi(s))-\widetilde{B}(\Phi(\lambda(1-\pi(s)))))}{\widetilde{A}(\lambda)(1-\pi(s))\widetilde{B}(\Phi(\lambda(1-\pi(s)))-\pi(s)(1-\widetilde{B}(\Phi(\lambda(1-\pi(s)))))}. \quad (40)$$

The corresponding expected values are given by

$$\mathrm{E}W = \frac{\lambda^2(\theta\mu\gamma_2-2)\beta_1^3+2\rho(\lambda+\theta-\rho\theta)\beta_1^2+\rho^2\theta\beta_2}{2\lambda\theta(\widetilde{A}(\lambda)-\rho)\beta_1^2}, \quad (41)$$

$$\mathrm{E}W_q = \frac{\lambda^2(\theta\mu\gamma_2-2)\beta_1^3+2\rho(\lambda+\theta-\widetilde{A}(\lambda)\theta)\beta_1^2+\rho^2\theta\beta_2}{2\lambda\theta(\widetilde{A}(\lambda)-\rho)\beta_1^2}, \quad (42)$$

where

$$\pi(s) \equiv \frac{(s+\lambda)\widetilde{A}(s+\lambda)\widetilde{B}(\Phi(s))}{s+\lambda-\lambda(1-\widetilde{A}(s+\lambda))\widetilde{B}(\Phi(s))}. \quad (43)$$

Proof. Note that the system is empty or the server is idle, busy, or under repair when a primary customer arrives. Thus, we have

$$\widetilde{W}(s) = p_{00}\mathrm{E}(\mathrm{e}^{-sv} \mid C=0, Q=0) + \sum_{j=1}^{\infty}\int_0^{\infty} p_{0,j}(v)\mathrm{E}(\mathrm{e}^{-sW} \mid C=0, Q=j, \xi_0=v)\mathrm{d}v$$
$$+ \sum_{j=0}^{\infty}\Big[\int_0^{\infty} p_{1,j}(x)\mathrm{E}(\mathrm{e}^{-sW} \mid C=1, Q=j, \xi_1=x)\mathrm{d}x + \int_0^{\infty}\int_0^{\infty} p_{2,j}(x,y)\mathrm{E}(\mathrm{e}^{-sW} \mid C=2, Q=j, \xi_1=x, \xi_2=y)\mathrm{d}x\mathrm{d}y\Big], \quad (44)$$

where C, Q, ξ_0, ξ_1, and ξ_2 denote, respectively, the server state, the number of customers in the retrial queue, the elapsed retrial time, the elapsed service time, and the elapsed repair time when the primary customer arrives.

It can be seen that W coincides with a generalized service time if the system is empty or the server is idle when the primary customer arrives. In this case, according to (3), $\mathrm{E}(\mathrm{e}^{-sW} \mid C=0, Q=0) = \mathrm{E}(\mathrm{e}^{-sW} \mid C=0, Q=j, \xi_0=v) = \widetilde{B}(\Phi(s))$. If j customers are already in the retrial queue and the server is busy, or under repair when the primary customer arrives, then W is equal to $W^*+W^{(j+1)}$, where W^* represents the waiting time of the customer being served spends in the system from the instant the primary customer arrives, and $W^{(j+1)}$ represents the total waiting time the $(j+1)$ customers in the retrial queue spend in the system from the moment the customer being served leaves the system. Since W^* and $W^{(i+1)}$ are independent, we get

$$\mathrm{E}(\mathrm{e}^{-sW} \mid C=1, Q=j, \xi_1=x) = \mathrm{E}\mathrm{e}^{-sW^{(j+1)}} \cdot \mathrm{E}(\mathrm{e}^{-sW^*} \mid C=1, Q=j, \xi_1=x), \quad (45)$$

$$\mathrm{E}(\mathrm{e}^{-sW} \mid C=2, Q=j, \xi_1=x, \xi_2=y) = \mathrm{E}\mathrm{e}^{-sW^{(j+1)}} \cdot \mathrm{E}(\mathrm{e}^{-sW^*} \mid C=2, Q=j, \xi_1=x, \xi_2=y). \quad (46)$$

In order to get $\mathrm{E}(\mathrm{e}^{-sW^*} \mid C=1, Q=j, \xi_1=x)$, $\mathrm{E}(\mathrm{e}^{-sW^*} \mid C=2, Q=j, \xi_1=x, \xi_2=y)$, we employ the following well-known formulas

$$P(y<\xi_1^{(r)}<y+\mathrm{d}y \mid \xi_1^{(*)}>x) = \frac{b(x+y)\mathrm{d}y}{1-B(x)}, \quad (47)$$

$$P(y<\xi_2^{(r)}<y+\mathrm{d}y \mid \xi_2^{(*)}>x) = \frac{g(x+y)\mathrm{d}y}{1-G(x)}, \quad (48)$$

where $\xi_1^{(*)}$ and $\xi_2^{(*)}$ represent the service time and repair time, respectively; $\xi_1^{(r)}$ and $\xi_2^{(r)}$ represent the remaining service time and remaining repair time at the instant the primary customer arrives, respectively. Thus, we have

$$\mathrm{E}(\mathrm{e}^{-sW^*} \mid C=1, Q=j, \xi_1=x) = \sum_{n=0}^{\infty}\frac{1}{1-B(x)}\int_0^{\infty} b(x+u) \cdot \mathrm{e}^{-su}\frac{(\mu u)^n}{n!}\mathrm{e}^{-\mu u}[\widetilde{G}(s)]^n\mathrm{d}u = \frac{1}{1-B(x)}\int_x^{\infty} b(u)\mathrm{e}^{-\Phi(s)(u-x)}\mathrm{d}u, \quad (49)$$

$$\mathrm{E}(\mathrm{e}^{-sW^*} \mid C=2, Q=j, \xi_1=x, \xi_2=y) = \sum_{n=0}^{\infty}\frac{1}{(1-B(x))(1-G(y))} \cdot \int_0^{\infty}\int_0^{\infty} b(x+u)g(y+v)\mathrm{e}^{-s(u+v)} \cdot \frac{[\mu u\widetilde{G}(s)]^n}{n!}\mathrm{e}^{-\mu u}\mathrm{d}u\mathrm{d}v$$
$$= \frac{1}{(1-B(x))(1-G(y))} \cdot \int_x^{\infty}\int_y^{\infty} b(u)g(v)\mathrm{e}^{-\Phi(s)(u-x)-s(v-y)}\mathrm{d}u\mathrm{d}v. \quad (50)$$

In order to obtain $\widetilde{W}(s)$, it still needs to find $\mathrm{E}\mathrm{e}^{-sW^{(j+1)}}$. Firstly, we calculate $\mathrm{E}\mathrm{e}^{-sW^{(1)}}$. After a service completion there exists a competition for service between a primary customer and a customer in the retrial queue. Therefore,

$$\begin{aligned} Ee^{-sW^{(1)}} &= \sum_{n=0}^{\infty}\widetilde{B}(\Phi(s))\int_0^{\infty}\int_y^{\infty}e^{-sy}\lambda e^{-\lambda x}dx\,dA(y) \\ &\times\left[\widetilde{B}(\Phi(s))\int_0^{\infty}\int_0^{y}e^{-sx}\lambda e^{-\lambda x}dx\,dA(y)\right]^n \\ &= \sum_{n=0}^{\infty}\widetilde{A}(s+\lambda)\widetilde{B}(\Phi(s))\Big[\frac{\lambda}{s+\lambda} \\ &\cdot(1-\widetilde{A}(s+\lambda))\widetilde{B}(\Phi(s))\Big]^n \\ &= \frac{(s+\lambda)\widetilde{A}(s+\lambda)\widetilde{B}(\Phi(s))}{s+\lambda-\lambda(1-\widetilde{A}(s+\lambda))\widetilde{B}(\Phi(s))} \\ &\equiv \pi(s). \end{aligned} \tag{51}$$

Secondly, according to the description of our model, it can be seen that

$$Ee^{-sW^{(j+1)}} = (\pi(s))^{j+1}, \quad j = 0,1,\cdots. \tag{52}$$

With the help of results in Theorem 2, after substituting (52), (49), (50) into the corresponding expressions above, and then arranging these new expressions into (44), we obtain $\widetilde{W}(s)$. Finally, noting that

$$\widetilde{W}(s) = \widetilde{B}(\Phi(\lambda(1-s))\widetilde{W}_q(s),$$

we can get the expression for $\widetilde{W}_q(s)$. This completes the proof.

4 Reliability indexes of the server

We now consider some reliability quantities of the server in this section.

Define $A(t) = P\{$the service station is up at time $t\}$ as the point-wise availability of the server, and define the steady-state availability of the server as

$$A = \lim_{t\to\infty}A(t).$$

Theorem 4. The steady-state availability of the server is $A = 1-\rho$.

Proof. This is readily obtained by considering the following equation

$$A = p_{00} + \lim_{z\to 1}\left[\int_0^{\infty}P_0(z,v)dv + \int_0^{\infty}P_1(z,x)dx\right],$$

together with (11) and (12). Q.E.D.

Theorem 5. The steady-state failure frequency of the server is $W_f = \mu\lambda\beta_1$.

Proof. Since the steady-state failure frequency of the service station is

$$W_f = \sum_{j=0}^{\infty}\int_0^{+\infty}\mu p_{1,j}(x)dx,$$

we get $W_f = \lim_{z\to 1}\int_0^{+\infty}\mu P_1(z,x)dx = \mu\lambda\beta_1$.

Q.E.D.

Denote by τ the time to the first failure of the server. Then the reliability function of the server is

$$R(t) = P(\tau > t).$$

Theorem 6. The Laplace-Stieltjes transform of $R(t)$ is given by

$$\begin{aligned} R^*(s) &= \frac{1}{s+\lambda}[1+H_0(s)] \\ &+ \frac{H_1(s)}{s+\lambda}[1-\widetilde{A}(s+\lambda)] \\ &+ \frac{H_2(s)}{s+\mu}[1-\widetilde{B}(s+\mu)], \end{aligned} \tag{53}$$

where

$$H_0(s) = \frac{\lambda\omega(s)}{(s+\lambda-\lambda\omega(s))\widetilde{A}(s+\lambda)},$$

$$H_1(s) = \frac{\lambda(s+\lambda)\omega(s) - \lambda\widetilde{B}(s+\mu)[(s+\lambda-\lambda\omega(s))\widetilde{A}(s+\lambda)+\lambda\omega(s)]}{(s+\lambda-\lambda\omega(s))\widetilde{A}(s+\lambda)[(s+\lambda)-(\lambda+s\widetilde{A}(s+\lambda))\widetilde{B}(s+\mu)]},$$

$$H_2(s) = \frac{\lambda(s+\lambda)[1-\omega(s)]}{(s+\lambda-\lambda\omega(s))[(s+\lambda)-(\lambda+s\widetilde{A}(s+\lambda))\widetilde{B}(s+\mu)]},$$

and $\omega(s)$ is the minimum absolute value root of the equation

$$\begin{aligned} x &= \frac{1}{s+\lambda}\{[\lambda x + (s+\lambda-\lambda x)\widetilde{A}(s+\lambda)] \\ &\cdot\widetilde{B}(s+\mu+\lambda-\lambda x)\} \end{aligned} \tag{54}$$

inside $|x| = 1$, $\mathrm{Re}(s) > 0$.

Proof. In order to find the reliability of the server, let the failure states of the server be absorbing states. Then we obtain a new system. In the new system, we use the same notations as in the previous section. Then we can get the following set of equations:

$$\frac{d}{dt}p_{00}(t) + \lambda p_{00}(t) = \int_0^{\infty}p_{1,0}(t,x)\beta(x)dx, \tag{55}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial v} + \lambda + \alpha(v)\right]p_{0,j}(t,v) = 0, \quad j \geqslant 1, \tag{56}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \mu + \beta(x)\right]p_{1,j}(t,x) = (1-\delta_{0j})\lambda p_{1,j-1}(t,x), \quad j \geqslant 0. \tag{57}$$

The boundary conditions are

$$p_{0,j}(t,0) = \int_0^\infty p_{1,j}(t,x)\beta(x)\mathrm{d}x, \quad (58)$$

$$p_{1,j}(t,0) = \lambda\Big[\delta_{0j}p_0(t) + (1-\delta_{0j})\int_0^\infty p_{0,j}(t,v)\mathrm{d}v\Big] + \int_0^\infty p_{0,j+1}(t,v)\alpha(v)\mathrm{d}v, \quad (59)$$

with the initial condition: $p_{00}(0)=1$.

By taking Laplace transforms of these equations, we obtain

$$sp_{00}^*(s) - 1 = -\lambda p_{00}^*(s) + \int_0^\infty \beta(x)p_{1,0}^*(s,x)\mathrm{d}x, \quad (60)$$

$$sp_{0,j}^*(s,v) + \frac{\partial p_{0,j}^*(s,v)}{\partial v} = -(\lambda+\alpha(v))p_{0,j}^*(s,v), \quad j\geqslant 1, \quad (61)$$

$$sp_{1,j}^*(s,x) + \frac{\partial p_{1,j}^*(s,x)}{\partial x} = -(\lambda+\mu+\beta(x))p_{1,j}^*(s,x) + (1-\delta_{0j})\lambda p_{1,j-1}^*(s,x), \quad j\geqslant 0, \quad (62)$$

$$p_{0,j}^*(s,0) = \int_0^\infty \beta(x)p_{1,j}^*(s,x)\mathrm{d}x, \quad (63)$$

$$p_{1,j}^*(s,0) = \lambda\delta_{0,j}p_{00}^*(s) + \lambda(1-\delta_{0j})\cdot\int_0^\infty \beta(x)p_{0,j}^*(s,v)\mathrm{d}v + \int_0^\infty \beta(x)p_{0,j+1}^*(s,v)\alpha(v)\mathrm{d}v, \quad j\geqslant 0. \quad (64)$$

Define the following generating functions

$$\phi_0(z,s,v) = \sum_{j=1}^\infty p_{0,j}^*(s,v)z^j,$$

$$\phi_1(z,s,x) = \sum_{j=0}^\infty p_{1,j}^*(s,x)z^j.$$

Multiplying Eqs. (60)—(64) by z^j and summing over j, we obtain the following basic equations after some algebraic manipulations:

$$p_{00}^*(s) = \frac{1}{s+\lambda}\Big[1 + \int_0^\infty \beta(x)\phi_1(0,s,x)\mathrm{d}x\Big], \quad (65)$$

$$\frac{\partial\phi_0(z,s,v)}{\partial v} = -(s+\lambda+\alpha(v)\phi_0(z,s,v)), \quad (66)$$

$$\frac{\partial\phi_1(z,s,x)}{\partial x} = -(s+\lambda-\lambda z+\mu+\beta(x))\phi_1(z,s,x), \quad (67)$$

$$\phi_0(z,s,0) = \int_0^\infty \beta(x)\phi_1(z,s,x)\mathrm{d}x - \int_0^\infty \beta(x)\phi_1(0,s,x)\mathrm{d}x, \quad (68)$$

$$\phi_1(z,s,0) = \lambda\Big[p_{00}^*(s) + \int_0^\infty \phi_0(z,s,x)\mathrm{d}x\Big] + \frac{1}{z}\int_0^\infty \alpha(v)\phi_0(z,s,v)\mathrm{d}v. \quad (69)$$

Solving (66), (67), we get

$$\phi_0(z,s,v) = \phi_0(z,s,0)\exp\{-(s+\lambda)v\}(1-A(v)), \quad (70)$$

$$\phi_1(z,s,x) = \phi_1(z,s,0)\exp\{-(s+\mu+\lambda-\lambda z)x\}\cdot(1-B(x)). \quad (71)$$

Substituting (70), (71) into (68) and (69), and making use of (65), we obtain

$$\phi_0(z,s,0) = \phi_1(z,s,0)\tilde{B}(s+\mu+\lambda(1-z)) + 1 - (s+\lambda)p_{00}^*(s), \quad (72)$$

$$\phi_1(z,s,0) = \lambda p_{00}^*(s) + \frac{\lambda z + [\lambda(1-z)+s]\tilde{A}(s+\lambda)}{(s+\lambda)z}\phi_0(z,s,0), \quad (73)$$

from which we get

$$\phi_0(z,s,0) = \frac{z(s+\lambda)[s+\lambda-\lambda\tilde{B}(s+\mu+\lambda(1-z))]p_{00}^*(s) + z(s+\lambda)}{[\lambda z + (s+\lambda(1-z))\tilde{A}(s+\lambda)]\tilde{B}(s+\mu+\lambda(1-z)) - z(s+\lambda)}. \quad (74)$$

By Rouché's theorem[25], the denominator has exactly one zero $\omega(s)$ inside the unit circle, and it is also the zero point for the numerator of the above equation. Simple algebra shows that

$$p_{00}^*(s) = \frac{1}{s+\lambda}\Big[1 + \frac{\lambda\omega(s)}{(s+\lambda-\lambda\omega(s))\tilde{A}(s+\lambda)}\Big]. \quad (75)$$

Substituting (75) into (74) and then into (73), we obtain the expressions of $\phi_0(z,s,0)$ and $\phi_1(z,s,0)$. This gives

$$\phi_0(z,s,v) = \phi_0(z,s,0)\exp\{-(s+\lambda)v\}(1-A(v))$$

$$= \frac{z(s+\lambda-\lambda\tilde{B}(s+\mu+\lambda(1-z)))\left[1+\frac{\lambda\omega(s)}{(s+\lambda-\lambda\omega(s))\tilde{A}(s+\lambda)}\right]-z(s+\lambda)}{[\lambda z+(s+\lambda(1-z))\tilde{A}(s+\lambda)]\tilde{B}(s+\mu+\lambda(1-z))-z(s+\lambda)}$$
$$\times \exp\{-(s+\lambda)v\}(1-A(v)), \quad (76)$$

$$\phi_1(z,s,x) = \frac{\lambda(z-\omega(s))}{[\lambda z+(s+\lambda(1-z))\tilde{A}(s+\lambda)]\tilde{B}(s+\mu+\lambda(1-z))-z(s+\lambda)}$$
$$\times \exp\{-(s+\mu+\lambda-\lambda z)x\}(1-B(x)). \quad (77)$$

Since

$$R^*(s) = p_{00}^*(s) + \lim_{z\to 1-}\int_0^\infty \phi_0(z,s,v)dv + \lim_{z\to 1-}\int_0^\infty \phi_1(z,s,x)dx, \quad (78)$$

upon substitution, we obtain the formula (53). Q.E.D.

From Theorem 6 we obtain

Corollary 3. The mean time to the first failure ($MTTFF$) of the server is given by

$$MTTFF = \frac{1}{\mu} + \frac{1}{\lambda}\frac{\omega(0)(1-\tilde{A}(\lambda))[\tilde{A}(\lambda)\tilde{B}(\mu)+2(1-\tilde{B}(\mu))]}{(1-\omega(0))\tilde{A}(\lambda)(1-\tilde{B}(\mu))}$$
$$+ \frac{1}{\lambda}\frac{\tilde{A}(\lambda)[1-2\tilde{B}(\mu)+\tilde{A}(\lambda)\tilde{B}(\mu)]}{(1-\omega(0))\tilde{A}(\lambda)(1-\tilde{B}(\mu))}. \quad (79)$$

Proof. From (53) and the following equation

$$MTTFF = \int_0^\infty R(t)dt = R^*(s)|_{s=0},$$

we obtain (79). Q.E.D.

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